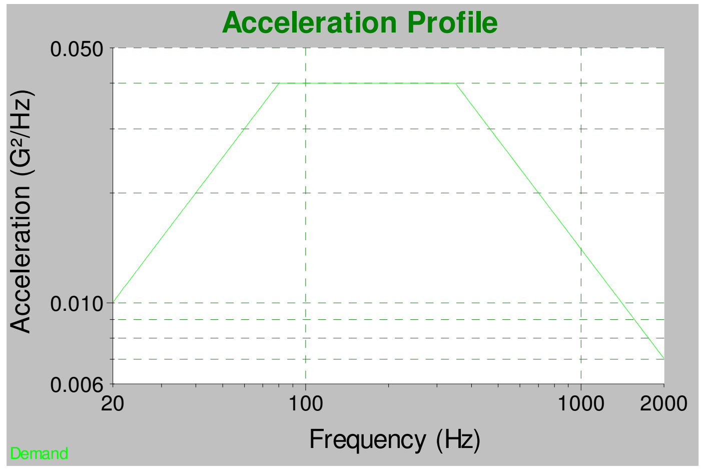
How to compute Random acceleration, velocity, and displacement values from a breakpoint table.

A random spectrum is defined as a set of frequency and amplitude breakpoints, like these:

Frequency **(Hz)**	**Amplitude** **(G^2/Hz)**
20	0.01005
80	0.04000
350	0.04000
2000	0.00704



To compute the RMS values from these breakpoints we need to compute the area under the curve defined by the breakpoints. At first glance this appears simple because the area can be split up into a group of squares and triangles, which are easy to compute. But note that the triangles are the result of straight lines on *log-log* graph paper, and not on *linear* graph paper. We can still take advantage of the triangles, however we need to use a special formula for computing the area of triangles on *log-log* graph paper.

The definition of a straight line on *log-log* graphs between two breakpoints (f_1,a_1) and (f_2,a_2) is a power relationship, where the slope is the exponent, and the offset is the multiplicative factor.

$$a = offset \cdot f^{\,slope} \tag{1}$$

The slope and offset that define this straight line, are computed as follows.

$$slope = \frac{\log(a_2) - \log(a_1)}{\log(f_2) - \log(f_1)} \tag{2}$$

$$offset = \frac{a_1}{f_1^{\,slope}} \tag{3}$$

Given this slope and offset we can integrate from f_1 to f_2 to compute the area under the line.

$$area = \frac{offset}{slope + 1}\left(f_2^{\,slope+1} - f_1^{\,slope+1}\right) \qquad \text{(if slope} \neq \text{-1)} \tag{4}$$

When *slope*=1 we have a special case where this formula doesn't hold. For this case we note that $a = offset / f$ which integrates to a *natural log* function. (Hint: some programs, including Microsoft Excel™ define the log() function as a base-10 logarithm, and define the ln() function as the natural (base-e) logarithm. Be sure to use the correct function in your calculation. As a test, for a natural logarithm, log(2.71828182845905) = 1.0.)

$$area = offset \cdot (\log(f_2) - \log(f_1)) \qquad \text{(if slope = -1)} \tag{5}$$

So, for each pair of breakpoints we can use equations (4) or (5) to compute the area under the curve. The total area under the curve will then be the sum of the individual area calculations between each pair of breakpoints, and this sum is the *mean-square* acceleration. We take the square-root of the result to get the RMS acceleration level. Using our example breakpoints, the sum is computed as follows:

Frequency (Hz)	**Amplitude (G^2/Hz)**	**Slope**	**Offset**	**Area (G^2)**
20	0.01005			
80	0.04000	0.9964	0.000508	1.50
350	0.04000	0	0.04000	10.80
2000	0.00704	-0.9967	13.73	24.47
			Total	36.77 G^2

$$\sqrt{36.77\ \mathrm{G}^2} = 6.064\ \mathrm{G}$$

Adding up the area values gives the mean-square acceleration of 36.77 G^2, and then taking the square root of this result gives the overall RMS value as ***6.064 G RMS***. Note that the acceleration units of this result will be the square root of the acceleration density units. When using density units of $(m/s^2)^2$/Hz, the result will be in m/s^2. When using density units of m^2/s^3 (which is simply a reduced form of the $(m/s^2)^2$/Hz unit), the result will be in m/s^2.

How about velocity?

The RMS velocity can be computed in the same manner; however the breakpoint numbers need to be converted from acceleration squared/Hz units to velocity squared/Hz units, with appropriate unit conversion, if required. This conversion is done using equation (6) which defines the relationship between velocity and acceleration for a sine wave of a given frequency.

$$velocity = \frac{acceleration}{2\pi f} \tag{6}$$

As a result, the equation for the lines connecting the breakpoints, in *velocity* density, becomes

$$v = \frac{offset}{(2\pi)^2} \cdot f^{(slope-2)} \tag{7}$$

This can be integrated from f_1 to f_2 to get the area under the velocity line.

$$area = \frac{1}{(2\pi)^2}\frac{offset}{(slope-1)}\left(f_2^{\,slope-1} - f_1^{\,slope-1}\right) \qquad (slope \neq 1) \tag{8}$$

When *slope* = 1 we need to use a *natural log* function.

$$area = \frac{offset}{(2\pi)^2} \cdot (\log(f_2) - \log(f_1)) \qquad (slope = 1) \tag{9}$$

Then we can sum the areas as before to get the *mean-square* velocity, and take the square root to get an RMS velocity value for the random spectrum. Also, when using acceleration units in G's, you also need to apply a conversion factor to get a suitable velocity unit. Common conversions are to convert G's to inches/s^2, or to m/s^2.

$$1 \text{ G} = 386.09 \text{ in/s}^2 = 9.80665 \text{ m/s}^2 \tag{10}$$

Using the previous example, the RMS velocity is computed as follows:

Frequency (Hz)	**Amplitude (G^2/Hz)**	**Slope**	**Offset**	**Area ($(G\cdot s)^2$)**
20	0.01005			
80	0.04000	0.9964	0.000508	1.760e-5
350	0.04000	0	0.04000	0.977e-5
2000	0.00704	-0.9967	13.73	0.141e-5
			Total	2.878e-5

$$\sqrt{2.878\text{e-}5\,(\text{G}\cdot\text{s})^2} = 0.005364 \text{ G}\cdot\text{s}\, RMS$$

Applying the unit conversion, we get

$$0.005364\,\text{G}\cdot\text{s}\cdot\frac{386.09\,\text{in/s}^2}{1\text{G}} = 2.07\,\text{in/s}\, RMS$$

How about Displacement?

The RMS displacement can be computed in the same manner; however the breakpoint numbers need to be converted from acceleration squared/Hz units to displacement squared/Hz units, with appropriate unit conversion, if required. This conversion is done using equation (11) which defines the relationship between acceleration and displacement for a sine wave of a given frequency.

$$displacement = \frac{acceleration}{(2\pi f)^2} \tag{11}$$

As a result, the equation for the lines connecting the breakpoints, in *displacement* density, becomes:

$$d = \frac{offset}{(2\pi)^4}\cdot f^{(slope-4)} \tag{12}$$

Now we can integrate this from f_1 to f_2 to get the area under the displacement line.

$$area = \frac{1}{(2\pi)^4}\frac{offset}{(slope-3)}\left(f_2^{slope-3} - f_1^{slope-3}\right) \quad (slope \neq 3) \tag{13}$$

When *slope*=3 we need to use a *natural log* function:

$$area = \frac{offset}{(2\pi)^4}\cdot\left(\log(f_2) - \log(f_1)\right) \quad (slope = 3) \tag{14}$$

Then we can sum the areas as before to get the *mean-square* displacement, and take the square root to get an RMS displacement value for the random spectrum. When using acceleration units in G's, you also need to apply a conversion factor such as defined in equation (10) to get a suitable displacement unit.

Using the previous example, the RMS displacement is computed as follows:

Frequency (Hz)	**Amplitude (G^2/Hz)**	**Slope**	**Offset**	**Area ($(G \cdot s^2)^2$)**
20	0.01005			
80	0.04000	0.9964	0.000508	3.773e-10
350	0.04000	0	0.04000	0.165e-10
2000	0.00704	-0.9967	13.73	0.0015e-10
			Total	3.994e-10

$$\sqrt{3.994\text{e-}10\left(\text{G}\cdot\text{s}^2\right)^2} = 1.985\text{e-}5\ \text{G}\cdot\text{s}^2\ \mathit{RMS}$$

Applying the unit conversion, we get

$$1.985\text{e-}5\ \text{G}\cdot\text{s}^2 \cdot \frac{386.09\,\text{in/s}^2}{1\text{G}} = 0.0077\,\text{in}\ \mathit{RMS}$$

But you are more likely interested in ***peak-to-peak displacement*** instead of the RMS displacement value, because your shaker travel limits are rated as a peak-to-peak value. Since the vibration is Gaussian random it is not possible to find an absolute peak value. What we can compute is an average, or typical, peak value. For Gaussian random values, the average one-sided peak level is about 3 times the RMS value (also called the 3-sigma level). To get double-sided displacement, a.k.a. peak-to-peak displacement, this number is doubled. So the *typical* peak-to-peak value is computed as:

Typical peak-peak displacement = 2 · 3 · (0.0077 in *RMS*) = 0.046 in *peak-to-peak*

Now it is important to know that this is simply the *typical* peak-to-peak displacement value. This peak value will occur relatively frequently during your test, so you will often see peaks at this level. But you will also have occasional peaks at levels higher than 3·RMS. How high those peaks get can only be defined in terms of probabilities, so the appropriate question is not how high the peaks are, but how often the peaks will be that high. For Gaussian random data, the amplitudes will be above 3·RMS 0.27% of the time. They will be above 4·RMS 0.006% of the time. And there is still a 0.0000002% chance that you can get a peak greater than 6·RMS. This is very small probability, but not impossible! *In practice you should have 30% to 50% of headroom above the typical peak-to-peak displacement value for the occasional higher peak levels.*

How about Sine-on-Random?

Sine-on-Random tests have a background random vibration, with one or more sine tones added over top of the background vibration. The background random levels are computed the same as for standard random, as described above. The sine tones then add to the random *in the mean-square*, which simply means you take the all of the RMS levels, square them, add them together, and take the square root of the result to get the overall RMS value. Since the sine tones are usually defined in terms of their peak acceleration, they need to first be converted from peak to RMS, which for a sine tone is as simple as dividing the peak value by $\sqrt{2}$.

Sine-on-Random Example: Use the same background random as defined above, and add in sine tones of 1.0G peak at 50 Hz, 2.0 G peak at 80 Hz, and 1.5 G peak at 110 Hz.

Frequency (Hz)	**Acceleration G Peak**	**Acceleration G RMS**	**Squared G^2**
random	-	6.064	36.77
50	1.0	0.707	0.50
80	2.0	1.414	2.00
110	1.5	1.061	1.13
		Total	40.40

$$\sqrt{40.40\,\mathrm{G}^2} = 6.36\,\mathrm{G}\,RMS$$

Velocity works the same way, remembering to convert the acceleration to velocity using equation (6), and then converting the result to the appropriate velocity unit. When summing the squared values, be sure the units for the background random and the sine tones match.

Frequency (Hz)	**Acceleration G Peak**	**Acceleration G RMS**	**Velocity G·s RMS**	**Squared $(G\cdot s)^2$**
random	-	-	0.005364	2.878e-5
50	1.0	0.707	0.002251	0.507e-5
80	2.0	1.414	0.002813	0.792e-5
110	1.5	1.061	0.001535	0.236e-5
			Total	4.411e-5

$$\sqrt{4.411\text{e-5}\ (\mathrm{G}\cdot\mathrm{s})^2} = 0.006642\,\mathrm{G}\cdot\mathrm{s}\,RMS$$

Applying the unit conversion, we get

$$0.006642\,\mathrm{G}\cdot\mathrm{s}\cdot\frac{386.09\,\mathrm{in/s}^2}{1\mathrm{G}} = 2.56\,\mathrm{in/s}\,RMS$$

Displacement would work the same way if we were interested in *RMS* values. To get peak displacement, however, we note that the *peak-to-peak* sine displacement will be $2\sqrt{2}$ times the *RMS* displacement for the sine tones, so the *RMS*-to-(*peak-to-peak*) conversion factor for 3-sigma random peaks, as assumed for the random background vibration, will not apply to the sine tones. To get the overall *peak-to-peak* displacement requirements it is better to assume that the peaks for the sine tones and the random background could all occur at the same point in time, so the peak displacement values will simply add together. So, to get the overall Sine-on-Random *peak-to-peak* displacement requirement, we add up the *peak-to-peak* displacements for the random background plus each of the sine tones. Remember we convert acceleration to displacement using equation (11) and then convert the displacement units using equation (10).

Frequency (Hz)	**Acceleration (G pk)**	**Displacement ($G \cdot s^2$ pk-pk)**	**Displacement in pk-pk**
random	-	-	0.046
50	1.0	1.433e-5	0.006
80	2.0	0.112e-5	0.004
110	1.5	0.444e-5	0.002
		Total	0.058 in *peak-to-peak*